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BIB(8, 56, 21, 3, 6) and BIB(10, 30, 9, 3, 2) Designs with Repeated Blocks*

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If there are less than b distinct blocks in a BIB design with b blocks then we say the design has repeated blocks. The set of distinct blocks of a design is called the support of the design. BIB designs with repeated blocks, besides being optimal, have special applications in the design of experiments and controlled samplings. Construction of BIB(v, b, r, k, λ) designs with repeated blocks becomes complicated whenever the three parameters b, r, and λ are relatively prime. BIB(8, 56, 21, 3, 6) designs are examples of such designs with the *smallest number of varieties*. BIB(10, 30, 9, 3, 2) designs are such designs with the *smallest number of blocks*. We make an interesting observation about BIB(8, 56, 21, 3, 6) designs and give a table of such designs with 30 different support sizes. We prove, by construction, that a BIB(10, 30, 9, 3, 2) design exists if and only if the support size belongs to $\{21, 23, 24, 25, 26, 27, 28, 29, 30\}$. Other results are also given.

1. INTRODUCTION

The customary definition of a balanced incomplete block design (BIB) does not require that all the blocks of the design be distinct. The set of all distinct blocks of a BIB design is called the *support* of the design and the cardinality of the support is denoted by b^* and is referred to as the *support size* of the design.

Throughout this paper we shall use the notation BIB$(v, b, r, k, \lambda \mid b^*)$ to indicate a BIB design with support size b^* based on v varieties in b blocks of size k such that each variety appears in r blocks and each pair of varieties appears in λ blocks. The notation for a block of size k consisting of the varieties $x_1, x_2, \ldots, x_k$ will be $(x_1 x_2 \cdots x_k)$ or simply $x_1 x_2 \cdots x_k$, while the order among the k varieties is immaterial. Clearly in a BIB$(v, b, r, k, \lambda \mid b^*)$,

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$b^* = b$ if $\lambda = 1$. The parameter λ has to be greater than one if the design is to have $b^* < b$. BIB designs with repeated blocks (i.e., $b^* < b$) have useful practical applications in design of experiments and controlled sampling from finite populations as explained in Foody and Hedayat [2], Hedayat [4] and Wynn [12]. For the purpose of studying BIB designs in which at least one block is repeated two or more times it is helpful if we divide the collection of all BIB designs with $\lambda \geqslant 2$ into three mutually exclusive and exhaustive families.

Family 1

This family consists of those BIB(v, b, r, k, λ) designs whose parameters b, r, and λ have a common integer divisor, $t > 1$, and there exists one or more BIB$(v, b/t, r/t, k, \lambda/t)$ designs. One way to produce a member of this family with repeated blocks is to collect t designs of the latter type with two or more common blocks. Another way is to take t copies of a design of the latter type. An example of such a design is BIB(7, 14, 6, 3, 2) design. To produce such a desoign with repeated blocks we can take either the 14 blocks of the two BIB designs or duplicating one of the two BIB(7, 7, 3, 3, 1) designs

346	235		346	123
156	457		156	147
267	137	,	267	357
124			245	

Family 2

This family consists of all BIB(v, b, r, k, λ) designs whose parameters b, r, and λ have one or more common integer divisors greater than one but there is no BIB$(v, b/t, r/t, k, \lambda/t)$ design if $t > 1$ is one of the divisors of b, r, and λ. Therefore no member of this family with repeated blocks can be generated by collecting or taking copies of smaller size BIB designs of the type BIB$(v, b/t, r/t, k, \lambda/t)$. The parameters of typical examples of such BIB designs are

v	b	r	k	λ	t
15	42	14	5	4	2
36	84	14	6	2	2
43	86	14	7	2	2

BIB designs with these parameters exist, but they cannot be generated by collecting or copying their smaller size counterpart designs whose parameters are

v	$b/2$	$r/2$	k	$\lambda/2$
15	21	7	5	2
36	42	7	6	1
43	43	7	7	1

It is well known that BIB designs of the latter types do not exist. Therefore to produce BIB designs with repeated blocks of the former types we have to use other methods, such as the trade off technique of Hedayat and Li [5, 6] and Hedayat and Hwang [7].

Family 3

The parameters v, b, r, k, λ for the members of this family are such that the greatest common integer divisor of b, r, and λ is one. Thus no member of this family with repeated blocks (or otherwise) can be generated by collecting or copying smaller size designs of the type BIB$(v, b/t, r/t, k, \lambda/t)$ designs. Therefore, in order to construct a member of this family with repeated blocks we have to resort to other techniques such as trade off. van Lint [10] has obtained some interesting results concerning members of this family.

Our main purpose in this paper is to study in depth two special types of BIB designs in Family 3. Before we identify the parameters of these designs we need to point out some useful facts about BIB designs with repeated blocks in general, and for members of Family 3 in particular. Mann [8] proved that if f blocks of a BIB design are identical, then

$$b \geqslant fv \qquad \text{or} \qquad f \leqslant b/v = r/k. \tag{1.1}$$

Note that the Mann inequality covers the Fisher inequality, $b \geqslant v$. van Lint and Ryser [11] gave a generalization of the Mann inequality and also proved that if the frequency f in (1.1) reaches the upper bound b/v then

$$f \text{ divides } (b, r, \lambda), \tag{1.2}$$

where (b, r, λ) denotes the greatest common integer divisor of b, r, and λ. Applying (1.1) and (1.2) to a BIB(v, b, r, k, λ) design with repeated blocks in Family 3 we see that for any such member

$$b > 2v \qquad \text{and} \qquad r > 2k. \tag{1.3}$$

Also, from the general necessary condition

$$\lambda \binom{v}{2} = b \binom{k}{2} \tag{1.4}$$

which must hold in any BIB design we observe that (see van Lint [10]):

$$v \neq 2k \text{ or } 2k+1 \text{ and } k>2 \tag{1.5}$$

for any BIB design with repeated blocks belonging to Family 3. Finally, since the complement of a BIB(v, b, r, k, λ) design is also a BIB design with block size $v-k$ we conclude that in our study of BIB designs with repeated blocks belonging to Family 3 we can restrict ourselves to those BIB designs in which

$$v \geqslant 8 \qquad \text{and} \qquad 3 \leqslant k \leqslant (v/2)-1. \tag{1.6}$$

Therefore $v=8$ is the minimum number of treatments in a design with repeated blocks belonging to Family 3. For $v=8$ then by (1.5) and (1.4) we conclude that other parameters are uniquely determined as $b=56$, $r=21$, $k=3$ and $\lambda=6$. Therefore, part of our effort here is to gather some new and known facts about BIB(8, 56, 21, 3, 6) designs since they belong to Family 3, and have the smallest number of treatments. We also study BIB(10, 30, 9, 3, 2) designs. These are precisely those designs with minimum number of blocks in Family 3 which admit repeated blocks. To prove that these designs have minimum number of blocks, it can be argued easily as follows. Due to Mann's inequality (1.1) or inequality (1.3), to show that $b=30$ is the smallest b for designs in Family 3 to have repeated blocks we need only to look at those designs in Family 3 with $8 \leqslant v \leqslant 14$ and $b>2v$. The following table comprises the parameters of all such BIB designs:

v	b	r	k	λ	v	b	r	k	λ
8	56	21	3	6	12	182	55	5	20
10	30	9	3	2	13	39	15	5	5
11	55	15	3	3	14	182	39	3	6
11	55	20	4	6	14	91	26	4	6
12	44	11	3	2	14	182	65	5	20
12	33	11	4	3	14	91	39	6	15

Clearly, this table supports our claim.

Having motivated our study we shall now briefly indicate the content of the paper. Concerning BIB(8, 56, 21, 3, 6) designs we have gathered the following facts. In Section 2 we prove that besides the trivial case $k=2$, the pair of parameters $v=8$ and $k=3$ is the only case where the basic necessary condition (1.4), or equivalently

$$bk=vr \qquad \text{and} \qquad \lambda(v-1)=r(k-1) \tag{1.7}$$

requires that we should take $\binom{v}{k}=\binom{8}{3}=56$ blocks, i.e., no reduced size BIB design is possible. Recall that for the given v and k we can form a BIB design by taking all $\binom{v}{k}$ subsets of size k based on v treatments. This will be an unreduced design and has no repeated blocks. While for $v=8$ and $k=3$ we cannot reduce the number of blocks, we can reduce the support size to as low as 22 and produce various BIB designs with repeated blocks. Based on Foody and Hedayat [2] for every $22 \leqslant b^* \leqslant 50$ and $b^*=52$ there is at least one BIB$(8, 56, 21, 3, 6 \mid b^*)$ design. Clearly these 30 designs are nonisomorphic. For completeness, a table of such design is also given. In Section 3 we have proved that a BIB$(10, 30, 9, 3, 2 \mid b^*)$ design exists if and only if $b^*=21$ or $23 \leqslant b^* \leqslant 30$. Table II gives an example for each such design. The uniqueness of the design with support size 21 is shown in Section 3. Also we have indicated that our design with support size 29 is nonisomorphic to the design of Parker [9].

2. Study of BIB(8, 56, 21, 3, 6) Designs with Repeated Blocks

It is easy to argue (see below) that for $v=8$ and $k=3$ we need at least 56 blocks to form a BIB design. One way to form such a design is to let the 56 blocks be all $\binom{8}{3}=56$ distinct subsets of size 3 based on 8 treatments. Clearly the resulting design has support size $b^*=b=56$ and thus lacks any repeated blocks. To produce a design with repeated blocks we have to reduce the support size. Thus a problem of interest in this case is to discover what support sizes are possible. As we shall see later, at least 30 different support sizes are possible. Before we discuss these various support sizes we shall make an interesting observation concerning all BIB designs with 8 treatments in blocks of size 3.

For $v > k \geqslant 2$ it is well known that a necessary condition for the existence of a BIB(v, b, r, k, λ) design is

$$\lambda \binom{v}{2} = b \binom{k}{2} \tag{2.1}$$

or equivalently

$$bk = vr \qquad \text{and} \qquad \lambda(v-1) = r(k-1). \tag{2.2}$$

A very useful version of (2.2) is given below.

Proposition 2.1. *For given v and k with $v > k \geqslant 2$, a necessary condition for the existence of a BIB(v, b, r, k, λ) design is that the triple $[b, r, \lambda]$ must be an integer multiple of $[\binom{v}{k}/d, \binom{v-1}{k-1}/d, \binom{v-2}{k-2}/d]$, where d is the greatest common divisor of $\binom{v-i}{k-i}$, $i=0, 1, 2$.*

Proof. From (2.2), $b = r(v/k) = r\binom{v}{k}/\binom{v-1}{k-1}$ and $\lambda = r(k-1)/(v-1) = r\binom{v-2}{k-2}/\binom{v-1}{k-1}$. Hence $b/\binom{v}{k} = r/\binom{v-1}{k-1} = \lambda/\binom{v-2}{k-2} = h$. Note that the common ratio h must be a rational number so that the numbers $b = \binom{v}{k}h$, $r = \binom{v-1}{k-1}h$ and $\lambda = \binom{v-2}{k-2}h$ are integers. Let d be the greatest common divisor of $\binom{v}{k}$, $\binom{v-1}{k-1}$ and $\binom{v-2}{k-2}$. Then $\binom{v}{k}/d$, $\binom{v-1}{k-1}/d$ and $\binom{v-2}{k-2}/d$ are relatively prime. In order that b, r, λ remain integer values, h must be equal to h'/d for some integer h', hence the result.

COROLLARY 2.1. *For given v and k with $v > k \geqslant 2$, let $b_{\min}$ be the minimum solution for b satisfying* (2.2). *Then $b_{\min} = \binom{v}{k}$ if and only if $\binom{v}{k}$, $\binom{v-1}{k-1}$ and $\binom{v-2}{k-2}$ are relatively prime.*

Corollary 2.1 indicates that we need at least $\binom{v}{k}$ blocks to form a BIB design if the number of varieties v and the block size k are such that $\binom{v}{k}$, $\binom{v-1}{k-1}$ and $\binom{v-2}{k-2}$ are relatively prime. Are there many values of v and k for which the above statement is true? Theorem 2.1 will characterize all such v and k. First we need a lemma.

LEMMA 2.1. *Let v and k be two positive integers such that $v \geqslant 2k$ and $k \geqslant 3$, then $\prod_{i=2}^{k-1}(v-i) > k!$ unless $k = 3$ and $v \leqslant 8$.*

Proof. *Case* (i). If $k = 3$, $\prod_{i=2}^{k-1}(v-i) = v-2$ and $k! = 3! = 6$. Thus $\prod_{i=2}^{k-1}(v-i) > k!$ unless $v \leqslant 8$.

Case (ii). If $k > 3$, then $v \geqslant 2k \geqslant 8$. For all i such that $2 \leqslant i \leqslant k-1$ we have $v - i \geqslant v - k + 1 \geqslant 2k - k + 1 = k + 1$. Moreover $v - 2 \geqslant 2k - 2 = 2(k-1)$ and $v - 4 \geqslant 2k - 4 = 2(k-2)$.

Hence

$$\begin{aligned}
\prod_{i=2}^{k-1}(v-i) &= (v-2)(v-3)(v-4)\cdots(v-k+1) \\
&\geqslant 2(k-1)\cdot k\cdot 2(k-2)\cdot \underbrace{k\cdots k}_{(k-5)\text{-factors}} \\
&> 2k(k-1)(k-2)\cdots 3\cdot 2\cdot 1 \\
&> k!
\end{aligned}$$

Since the complement of a BIB design based on v, b, k is another BIB design with v, b, $v-k$ then with no loss we limit our characterization to v and k such that $v \geqslant 2k$.

THEOREM 2.1. *For $v \geqslant 2k$ and $k \geqslant 2$, the only solutions for (v, k) such that $\binom{v}{k}$, $\binom{v-1}{k-1}$ and $\binom{v-2}{k-2}$ are relatively prime are $(v, k) = (v, 2)$ and* (8, 3).

Proof. The conclusion is obvious for any v whenever $k = 2$. Now let

$k \geqslant 3$. Since $\binom{v}{k} = \binom{v-2}{k-2} v(v-1)/[k(k-1)]$ the numbers $\binom{v}{k}$, $\binom{v-1}{k-1}$ and $\binom{v-2}{k-2}$ are relatively prime only if $\binom{v-2}{k-2}$ divides $k(k-1)$. This could possibly occur only if $\binom{v-2}{k-2} \leqslant k(k-1)$ which is equivalent to $\prod_{i=2}^{k-1}(v-i) \leqslant k!$. Lemma 2.1 implies that the possible v and k which satisfy this inequality are $(v, k) = (6, 3)$, $(7, 3)$ and $(8, 3)$. But it is easy to check that among these, $v = 8$ and $k = 3$ is the only pair with the desired property. This completes the proof of the theorem.

When the block size is 2 we have no choice but to take all pairs, $\binom{v}{2}$, to form a BIB design. Besides this trivial case $v = 8$ and $k = 3$ is the only pair for which the basic necessary condition (2.2) forces the number of blocks b to be at least $\binom{8}{3} = 56$ in order to form a BIB design. Note that we are not saying that for other value of v and k we can necessarily form BIB designs based on less than $\binom{v}{k}$ blocks (this is an interesting open problem). Other conditions besides (2.2) have to be utilized.

If we take $\binom{8}{3} = 56$ distinct subsets of size 3 based on 8 treatments we obtain a BIB(8, 56, 21, 3, 6) design with no repeated blocks. Note that even though we need 56 blocks to form a BIB design for the case $v = 8$ and $k = 3$, these 56 blocks do not have to be all distinct. Indeed in this case designs exist in which a block is repeated the maximum number of times $\lambda = 6$. Foody and Hedayat [2] have shown that BIB$(8, 56, 21, 3, 6 \mid b^*)$ designs with repeated blocks exist for any support size b^* as long as $b^* = 52$ or $22 \leqslant b^* \leqslant 50$. Such designs with repeated blocks with support sizes $b^* = 51$, 53, 54, 55 also exist. But for these support sizes we need at least 112 blocks to form a BIB design. Table I gives an example for each of the designs discussed. The construction of these designs are based on the same technique as what we use to construct BIB(10, 30, 9, 3, 2) designs as explained in Section 3.3. Note that for the same b, designs with different frequencies are necessarily nonisomorphic. Thus when $b = 56$, Table I contains 30 nonisomorphic BIB(8, 56, 21, 3, 6) designs with repeated blocks. Moreover, except for the case $b^* = 22$ and $b^* = 52$, the designs listed in Table I are not isomorphic to those designs given in the table of Foody and Hedayat [2] either. Many more such nonisomorphic designs can be constructed by the same technique or can be found in Hedayat [3].

3. Study of BIB(10, 30, 9, 3, 2) Designs with Repeated Blocks

As far as we know the existing literature of BIB designs contains two nonisomorphic BIB(10, 30, 9, 3, 2) designs. One design with support size 30, thus having no repeated blocks, was given by Fisher and Yates [1]. The other design with support size 29, thus containing two copies of a block, is due to Parker [9]. If we identify the 10 treatments by $\{a_1, b_1, c_1, d_1, e_1, a_2,$

TABLE I

BIB Designs with $v = 8$ and $k = 3$
Support Sizes 22 to 55

b^*	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55
b	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	112	56	112	112	112
123	–	–	–	–	1	–	1	1	–	1	1	–	–	1	2	1	–	2	2	1	–	–	–	1	–	1	–	1	1	2	–	2	2	3
124	6	6	6	6	5	6	5	5	6	5	4	6	5	5	4	5	5	3	4	4	4	4	3	2	3	2	3	–	–	2	2	2	2	2
125	–	–	–	–	–	–	–	–	–	–	1	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	–	1	2	1	1
126	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	1	1	1	1	2	3	1	3	3	3
127	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	1	1	1	1	1	2	1	3	1	1	2	1
128	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	–	1	1	1	1	1	1	1	1	1	1	2	1	2	2	2
134	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	2	1	1	1	1	1
135	–	5	5	5	5	5	5	3	1	1	1	2	2	2	2	2	2	2	1	2	3	3	2	3	3	2	2	1	–	1	2	1	1	1
136	–	1	–	1	–	1	–	2	3	2	2	2	2	–	–	2	2	–	1	1	1	1	1	–	1	1	1	2	1	2	1	2	2	2
137	3	–	–	–	–	–	–	–	–	–	–	–	–	1	1	–	1	1	1	1	1	1	2	1	1	1	1	–	1	4	1	4	4	4
138	3	–	1	–	–	–	–	–	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	1
145	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	1	–	–	1	–	2	2	2	–	2	2	2
146	–	–	–	–	1	–	1	1	–	1	2	–	1	1	1	–	–	2	1	1	–	1	–	2	1	1	1	1	–	2	1	2	2	2
147	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	2	1	2	2	2
148	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	1	1	1	1	1	1	2	2	2	2	1	1	1	3	1	3	3	3
156	3	–	1	1	1	1	1	1	1	1	–	1	–	1	1	1	1	1	1	1	1	1	2	1	1	1	1	–	1	3	1	3	3	3
157	3	–	–	–	–	–	–	2	4	4	4	3	3	3	3	3	3	3	3	3	1	2	1	2	2	2	2	1	1	2	1	2	2	2
158	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	1	1	1	4	1	2	3	3
167	–	3	3	2	2	2	2	–	–	–	–	1	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1
168	3	2	2	2	2	2	2	2	2	2	2	2	2	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1
178	–	3	3	4	4	4	4	4	2	2	2	2	2	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	–	1	2	1	2
234	–	–	–	–	–	–	–	–	–	–	1	–	1	–	–	–	–	–	–	–	1	1	1	1	1	1	–	2	1	3	1	3	3	3
235	1	–	–	–	–	–	–	2	2	2	1	1	–	–	–	1	1	1	1	1	1	1	1	1	1	1	2	–	2	3	1	1	2	2
236	2	3	3	2	2	2	2	–	–	–	–	1	1	3	3	1	2	2	2	2	2	2	2	2	2	2	2	1	–	3	2	3	3	2
237	2	3	3	3	3	3	3	3	3	3	3	3	3	1	–	2	2	1	–	1	1	1	1	1	1	–	1	1	1	–	1	2	1	1

238	1	–	–	1	–	1	–	–	1	–	–	1	1	1	1	1	1	–	1	1	1	1	1	–	1	1	1	1	1	1	1	1	1	1
245	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	1	1	1	1	1	2	1	2	2	2
246	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	2	2	–	2	2	2
247	–	–	–	–	–	–	–	–	–	–	–	–	–	1	2	1	1	2	2	2	1	1	1	1	1	2	1	1	1	1	1	1	1	1
248	–	–	–	–	1	–	1	1	–	1	1	–	–	–	–	–	–	1	–	–	–	–	–	1	–	–	1	1	1	2	1	2	2	2
256	–	2	2	2	2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	2	–	2	1	2	2	2
257	–	–	–	–	–	1	2	–	–	–	–	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	4	1	4	4	4
258	5	4	4	4	4	4	2	2	2	2	2	2	2	3	3	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1
267	4	1	1	2	2	2	–	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	–	1	–	1	–	–	1
268	–	–	–	–	–	1	2	2	2	2	2	2	2	–	–	2	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	2
278	–	2	2	1	1	–	1	1	1	1	1	1	1	2	2	1	1	1	2	1	1	1	1	1	1	1	1	1	1	4	1	4	4	4
345	4	–	–	–	–	–	–	–	2	2	2	2	2	2	2	2	2	2	2	2	1	1	2	1	1	1	1	1	–	2	1	2	2	2
346	2	1	1	1	1	1	1	1	1	1	–	1	–	1	1	1	1	1	1	1	1	1	1	1	1	1	1	–	1	2	1	2	2	2
347	–	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	2	2	2	2	1	1	2	1	2	2	2
348	–	3	3	3	3	3	3	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	2
356	–	–	–	–	–	–	–	–	–	–	1	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	2	1	–	1	1	1
357	1	–	–	–	–	–	–	–	–	–	–	–	–	1	1	–	–	–	1	–	–	–	–	–	–	1	–	2	1	2	1	2	2	2
358	–	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	3	1	5	4	4
367	–	–	1	1	1	1	1	1	1	1	1	1	1	1	1	1	–	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	2
368	2	1	1	1	2	1	2	2	1	2	2	1	1	1	1	1	1	2	1	1	1	1	1	2	1	1	1	1	1	2	1	2	2	3
378	–	1	–	–	–	–	–	–	–	–	–	–	–	–	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	–	1	1
456	2	2	2	2	2	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	2	2	2	2	1	1	2	2	2	2	2
457	–	4	4	4	4	3	2	2	–	–	–	–	–	–	–	–	–	–	–	–	1	1	1	1	1	–	1	–	1	2	1	2	2	2
458	–	–	–	–	–	–	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	2	–	1	1	1	1	1	1	2	1	2	2	2
467	1	–	–	–	–	–	2	2	2	2	2	2	2	1	1	2	2	1	1	1	1	1	2	1	1	1	1	2	1	3	1	3	3	3
468	1	3	3	3	2	2	–	–	1	–	–	1	1	1	1	1	1	–	1	1	2	1	2	–	1	1	1	1	1	1	1	1	1	1
478	5	–	–	–	–	1	–	–	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	2
567	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	2
568	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	1	1	1	1	1	2	1	2	2	2
578	1	–	1	1	1	1	1	1	1	1	1	1	1	–	–	1	1	1	–	1	2	1	2	1	1	1	1	1	1	–	1	–	–	–
678	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	–	1	1	1	1	–	1	–	1	1	1	1	1	1	4	1	4	4	3

Note. For each support size the number of blocks, b, is a minimum.

$b_2, c_2, d_2, e_2\}$ then the Fisher and Yates design can be constructed by cyclically permuting a, b, c, d, e according to the permutation $(abcde)$ of the six generating blocks

$$a_1 d_1 b_2, \quad b_2 c_1 b_1, \quad e_2 b_2 e_1, \quad e_2 a_2 b_2, \quad a_1 e_2 d_1, \quad b_1 c_1 e_2.$$

Thus the Fisher and Yates BIB(10, 30, 9, 3, 2) design is

$a_1 d_1 b_2$	$b_2 c_1 b_1$	$e_2 b_2 e_1$	$e_2 a_2 b_2$	$a_1 e_2 d_1$	$b_1 c_1 e_2$
$b_1 e_1 c_2$	$c_2 d_1 c_1$	$a_2 c_2 a_1$	$a_2 b_2 c_2$	$b_1 a_2 e_1$	$c_1 d_1 a_2$
$c_1 a_1 d_2$	$d_2 e_1 d_1$	$b_2 d_2 b_1$	$b_2 c_2 d_2$	$c_1 b_2 a_1$	$d_1 e_1 b_2$
$d_1 b_1 e_2$	$e_2 a_1 e_1$	$c_2 e_2 c_1$	$c_2 d_2 e_2$	$d_1 c_2 b_1$	$e_1 a_1 c_2$
$e_1 c_1 a_2$	$a_2 b_1 a_1$	$d_2 a_2 d_1$	$d_2 e_2 a_2$	$e_1 d_2 c_1$	$a_1 b_1 d_2$

The structure of Parker's design is best depicted if we identify the 10 treatments by $\{0, 1, 2, 3, 4, 5, 6, x, y, z\}$ and present the 30 blocks as

xyz	$x01$	$y02$	$z04$	124
xyz	$x12$	$y13$	$z15$	235
	$x23$	$y24$	$z26$	346
	$x34$	$y35$	$z30$	450
	$x45$	$y46$	$z41$	561
	$x56$	$y50$	$z52$	602
	$x60$	$y61$	$z63$	013

In the sequel we shall characterize all possible support sizes for BIB$(10, 30, 9, 3, 2 \mid b^*)$ designs. We shall prove that a BIB$(10, 30, 9, 3, 2 \mid b^*)$ exists if and only if $b^* \in \{21, 23, 24, 25, 26, 27, 28, 29, 30\}$. Thus no design with support size 22 or 20 and below is possible. For each support size we shall explicitly present a design. We also prove that the design with support size 21 is unique and our design of support size 29 is nonisomorphic to the Parker design.

DEFINITION 3.1. A BIB$(v, b, r, k, \lambda \mid b^*)$ design is said to be uniform on its support if each block in the support is repeated the same number of times in the design.

It is clear that if a BIB$(v, b, r, k, \lambda \mid b^*)$ is uniform on its support then the support itself is a BIB$(v, b^*, r^*, k, \lambda^*)$ design with $r^* = r/t$, and $\lambda^* = \lambda/t$ where $t = b/b^*$. Applying this observation to our case we shall obtain:

PROPOSITION 3.1. *Every BIB*(10, 30, 9, 3, 2) *design has at least* 16 *distinct blocks.*

Proof. Since $\lambda = 2$ the lowest possible value for the support size of a BIB(10, 30, 9, 3, 2) design is 15, which then forces the design to be uniform on its support. The conclusion in the paragraph below Definition 3.1 and the necessary conditions (2.2) rule out such possibility.

In order to rule out the existence of BIB(10, 30, 9, 3, 2) designs based on $b^* = 16, 17, 18, 19, 20, 22$ distinct blocks and establish their existence based on the remaining values of b^* we need to develop some tools which we shall now do.

Notation. Let D be a BIB(10, 30, 9, 3, 2) design. Since no blocks can be repeated more than twice in D we define

$$E_1 = \{B; \text{the block } B \text{ appears exactly once in } D\}$$

and

$$E_2 = \{\bar{B}; \text{the block } \bar{B} \text{ appears twice in } D\}.$$

Thus we shall write $D = E_1 \cup 2E_2$ and $D^* = E_1 \cup E_2$ where D^* denotes the support of D.

Let S be a subset of blocks of D. We define

$$r_s(x) \overset{\text{def.}}{=} \text{no. of blocks in } S \text{ which contain variety } x,$$

$$\lambda_s(xy) \overset{\text{def.}}{=} \begin{array}{l}\text{no. of blocks in } S \text{ which contain the pair}\\ \text{of varieties } x \text{ and } y.\end{array}$$

If we denote the set of 10 varieties by $V = \{0, 1, ..., 9\}$, then

$$\begin{aligned} r_D(x) = r_{E_1}(x) + 2r_{E_2}(x) = 9 \\ \lambda_{E_1}(xy) + 2\lambda_{E_2}(xy) = 2, \qquad \text{for all } x, y \text{ in } V. \end{aligned} \tag{3.1}$$

We observe that each pair of varieties appears either in E_1 or E_2 but not both, and when it appears in E_1, then it will appear in two blocks of E_1.

DEFINITION 3.2. A block $B = (xyz)$ in D is said to be of (3, 3, 3)-type if $r_{E_i}(x) = r_{E_i}(y) = r_{E_i}(z) = 3$ for $i = 1, 2$.

3.1. *Nonexistence of* BIB(10, 30, 9, 3, 2 | b^*) *Designs with* $b^* \leqslant 20$

By Proposition 3.1 there is no BIB(10, 30, 9, 3, 2 | b^*) with $b^* \leqslant 15$. We shall now rule out $b^* = 16$, 17, 18, 19 and 20. First we need some lemmas.

LEMMA 3.1. *If $D = E_1 \cup 2E_2$ is a BIB(10, 30, 9, 3, 2) design then each variety appears in at least 3 blocks of E_1.*

Proof. Let x be a variety in V. Since by (3.1), $r_{E_1}(x) + 2r_{E_2}(x) = 9$, thus $r_{E_1}(x)$ must be a nonzero odd integer, that is, x appears in at least one block of E_1. Let this block be (xyz). Since $\lambda = 2$, the pairs xy, xz cannot appear in E_2. Therefore there should exist two more blocks in E_1 to cover these pairs, consequently $r_{E_1}(x)$ is at least 3.

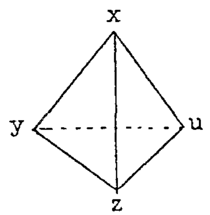
LEMMA 3.2. *Let $D = E_1 \cup 2E_2$ be a BIB(10, 30, 9, 3, 2) design. If $B_1 = \{xyz\}$ in E_1 is of (3, 3, 3)-type then E_1 contains (xyu), (xzu) and (yzu) for some variety u in V.*

Proof. By assumption $r_{E_1}(x) = r_{E_1}(y) = r_{E_1}(z) = 3$. Also by the observation we made below (3.1), $\lambda_{E_1}(xy) = \lambda_{E_1}(xz) = \lambda_{E_1}(yz) = 2$. We may assume the first four blocks of E_1 which covers twice the pairs xy, xz and yz are

$$B_1 = (xyz), \qquad B_2 = (xyu), \qquad B_3 = (xzv), \qquad B_4 = (yzw)$$

with u, v, w in $V - \{x, y, z\}$. Since the pairs xu, xv, yu, yw, zv and zw appear in blocks of E_1, thus they each should appear in two blocks of E_1. But there are no more blocks besides the above four blocks containing x, y, and z. This forces $u = v = w$.

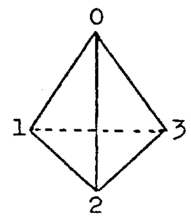
Remark. We may present the blocks of D by equilateral triangles with vertices labeled by varieties. Then a geometrical version of Lemma 3.2 is this: If E_1 contains (xyz) which is of (3, 3, 3)-type, then E_1 contains a pyramid based on x, y, z and u for some variety u:



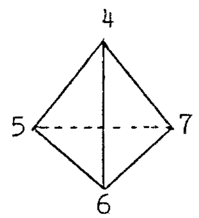
LEMMA 3.3. *If $D = E_1 \cup 2E_2$ is a BIB(10, 30, 9, 3, 2) design, then E_1 contains at least 12 blocks.*

Proof. From Lemma 3.1, E_1 must contain 10 blocks in order to cover each variety at least 3 times. If E_1 contains precisely 10 blocks then each block of E_1 is of (3, 3, 3)-type. We now argue that this cannot happen.

Suppose E_1 contains 10 blocks. Let $B_1 = (012)$ be in E_1. Then by Lemma 3.2 and the remark after that, E_1 contains the pyramid



the varieties 0, 1, 2 and 3 have already appeared 3 times in the four blocks of the above pyramid. We may now assume $B_5 = (456)$. Again E_1 contains the pyramid



and thus varieties 4, 5, 6 and 7 have appeared 3 times in the four blocks of the pyramid based on 4, 5, 6 and 7. Now we are left with varieties 8 and 9 *only* to be covered each three times in 2 blocks which is impossible. Finally, since $30 = |D| = |E_1| + 2|E_2|$ we conclude that the number of blocks in E_1 is even and by the above argument it is at least 12.

Utilizing the preceding results we can improve upon Proposition 3.1 and conclude that:

THEOREM 3.1. *Every BIB*(10, 30, 9, 3, 2) *design has at least* 21 *distinct blocks.*

Proof. If $D = E_1 \cup 2E_2$ is such a design, then

$$\begin{aligned} |E_1| + |E_2| &= b^*, \qquad \text{the support size} \\ |E_1| + 2|E_2| &= b = 30. \end{aligned} \tag{3.2}$$

By Lemma 3.3, $|E_1| \geqslant 12$ which implies that $|E_2| \leqslant 9$. Thus $b^* \geqslant \min |E_1| + \max |E_2| = 12 + 9 = 21$.

As we shall see in Section 3.3, BIB(10, 30, 9, 3, 2) designs based on 21, and 23 to 30 distinct blocks exist. In the following section we shall prove that such a design based on exactly 22 distinct blocks does not exist.

3.2. *Nonexistence of* BIB(10, 30, 9, 3, 2 | b^*) *Design with* $b^* = 22$

We shall prove a lemma:

LEMMA 3.4. *If $D = E_1 \cup 2E_2$ is a $BIB(10, 30, 9, 3, 2)$ design which has exactly 22 distinct blocks, then E_2 contains a block of (3, 3, 3)-type.*

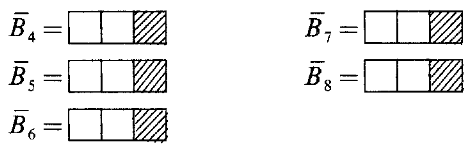
Proof. If $b^* = 22$ then from (3.2), $|E_1| = 14$ and $|E_2| = 8$. Thus we have $8 \times 3 = 24$ positions in 8 blocks of E_2 to be filled by at most 10 varieties. This implies that at least one variety has to appear in 3 blocks of E_2. Now assume the variety 0 appears in 3 blocks of E_2, say $\bar{B}_1$, $\bar{B}_2$ and $\bar{B}_3$. With no loss we may assume

$$\bar{B}_1 = (012), \qquad \bar{B}_2 = (034), \qquad \bar{B}_3 = (056).$$

By Lemma 3.1 the variety 0 cannot appear in any more blocks of E_2 and has to appear in 3 blocks of E_1. Since $\lambda = 2$ and 0 has to appear with varieties 7, 8 and 9 in two blocks of E_1 thus the first 3 blocks of E_1 are

$$B_1 = (078), \qquad B_2 = (079), \qquad B_3 = (089).$$

By (3.1), $\lambda_{E_2}(78) = \lambda_{E_2}(79) = \lambda_{E_2}(89) = 0$. Therefore each of the remaining blocks of E_2 contains at most one of 7, 8 and 9, that is only shaded cells in the following 5 blocks could be chosen for 7, 8 and 9:



The remaining 10 positions with two in each block are filled with varieties 1, 2, 3, 4, 5 and 6. It follows that at least four out of these six varieties has to appear twice in the remaining 5 blocks. Hence, at least one pair of (12), (34) and (56) in which each variety appears in two of these 5 blocks. We may assume it is (12). Therefore the block $\bar{B}_1 = (012)$ is of (3, 3, 3)-type.

THEOREM 3.2. *There is no $BIB(10, 30, 9, 3, 2)$ design with exactly 22 distinct blocks.*

Proof. Suppose $D = E_1 \cup 2E_2$ is a BIB(10, 30, 9, 3, 2) design with exactly 22 distinct blocks. From Lemma 3.4, the blocks in E_2 can be assumed to be

$$\begin{array}{lll} \bar{B}_1 = (012) & \bar{B}_4 = 1\,\square\,\square & \bar{B}_7 = 2\,\square\,\square \\ \bar{B}_2 = (034) & \bar{B}_5 = 1\,\square\,\square & \bar{B}_8 = \square\,\square\,\square \\ \bar{B}_3 = (056) & \bar{B}_6 = 2\,\square\,\square & \end{array}$$

and the first 9 blocks in E_1 are

$$\begin{array}{lll} B_1 = (078) & B_4 = 1\square\square & B_7 = 2\square\square \\ B_2 = (079) & B_5 = 1\square\square & B_8 = 2\square\square \\ B_3 = (089) & B_6 = 1\square\square & B_9 = 2\square\square \end{array}$$

Observe that $\bar{B}_8$ has to contain one variety chosen from 7, 8 and 9 and two varieties chosen from 3, 4, 5 and 6. This is because the pairs (78), (79), (89), (34) and (56) cannot appear together in $\bar{B}_8$. Thus with no loss of generality we may assume $\bar{B}_8 = (357)$. Note that none of the remaining blocks in $E_1 \cup E_2$ can contain the pair (35), (37) or (57). Moreover, by Lemma 3.1, each of the varieties 3 and 5 can appear in at most 3 blocks of E_2. Hence the only way to arrange $\bar{B}_i$, $i = 4, 5, 6, 7$, is

$$\begin{array}{ll} \bar{B}_4 = 13\square & \bar{B}_6 = 25\square \\ \bar{B}_5 = 17\square & \bar{B}_7 = 27\square \end{array} \tag{3.3}$$

or equivalently

$$\begin{array}{ll} \bar{B}_4 = 15\square & \bar{B}_6 = 23\square \\ \bar{B}_5 = 17\square & \bar{B}_7 = 27\square \end{array} \tag{3.4}$$

Without loss we may assume (3.3) is the case. Then each of the varieties 3, 5 and 7 appears in 3 blocks of E_2. Thus in E_1 we have

$$\begin{array}{lll} B_4 = (15x) & B_7 = (23z) & B_{10} = (5xy) \\ B_5 = (15y) & B_8 = (23u) & B_{11} = (3zu) \\ B_6 = (1xy) & B_9 = (2zu) & B_{12} = (789) \end{array}$$

with $x, y, z, u \in \{4, 6, 8, 9\}$ and $\{x, y\} \neq \{z, u\}$ since $\lambda = 2$. As far as we can see each pair of varieties has either not appeared or already appeared together in exactly 2 blocks of B_i, $1 \leqslant i \leqslant 12$. The only way to arrange the last 2 blocks of E_1 to fulfill the property that $\lambda_{E_1}(xy) = 0$ or 2 for all $x, y \in V$ is that $B_{13} = B_{14}$ which contradicts to our assumption on E_1. Therefore BIB(10, 30, 9, 3, 2) designs with exactly 22 distinct blocks do not exist.

3.3. *Existence and Construction of* BIB(10, 30, 9, 3, 2) *Designs with* 21 *and* 23 *to* 30 *Distinct Blocks*

We have already established that a BIB(10, 30, 9, 3, 2) design exists only if its support size b^* belongs to $\{21, 23 \leqslant b^* \leqslant 30\}$. The purpose of this section is to establish by construction, that all such designs exist. In Table II

TABLE II

BIB Designs with $v = 10$, $k = 3$ and $b = 30$

$b^* = 21$	$b^* = 23$	$b^* = 24$	$b^* = 25$	$b^* = 26$	$b^* = 27$	$b^* = 28$	$b^* = 29$	$b^* = 30$
012	013	013	015	013	015	016	016	014
013	015	017	016	016	016	018	017	018
023	023	023	023	024	024	024	024	025
045	024	024	024	025	025	025	025	026
046	046	046	034	034	034	034	034	034
056	056	056	056	056	036	036	036	036
078	078	058	078	078	078	057	058	057
079	079	079	079	079	079	079	079	079
089	089	089	089	089	089	089	089	089
123	123	123	123	123	123	123	123	123
2(147)	128	128	128	128	128	128	128	128
2(158)	2(147)	145	139	2(147)	138	135	135	135
2(169)	158	147	2(147)	158	2(147)	2(147)	147	147
2(248)	2(169)	158	158	159	159	159	148	159
2(259)	248	2(169)	169	169	169	169	159	167
2(267)	2(259)	248	248	239	239	239	169	169
2(349)	2(267)	2(259)	2(259)	248	248	248	239	239
2(357)	2(349)	2(267)	2(267)	259	259	259	248	247
2(368)	2(357)	2(349)	349	2(267)	2(267)	2(267)	259	248
456	2(368)	2(357)	2(357)	349	349	349	2(267)	259
789	456	2(368)	2(368)	2(357)	2(357)	357	349	267
	458	456	456	2(368)	368	368	357	349
	789	478	458	456	456	378	368	357
		789	469	458	458	456	378	368
			789	469	469	458	456	378
				789	568	469	457	456
					789	568	469	458
						789	568	469
							789	568
								789

Note. $2(xyz)$ means two copies of the block (xyz).

we have exhibited an example for each claimed design. For the benefit of the interested reader we shall briefly explain the technique which we used in constructing these designs. Our technique is based on the method of trade off of Hedayat and Li [5]. In an effort to conserve space, the reader is referred to Hedayat and Li [5] and Hedayat and Hwang [7] for details and concepts concerning the method of trade off. The basic idea is to trade some blocks with some other blocks without losing the BIB structure of the design. The resulting design may have larger or smaller support size than the original design. Hedayat and Li [5] have proved that the technique of trade off can be used to produce all BIB designs with various support sizes. For example, consider the following two sets of four blocks each:

I	II
xyz	uvw
xuv	yzw
yuw	xzv
zvw	xyu

By inspection it is easy to verify that a pair of varieties appear in a block of I if and only if it appears in a block of II. Such a set of blocks is called a $(v, 3)$ trade of volume 4 for any $v \geqslant 6$ since we have 6 varieties here and each set consists of 4 blocks. Now suppose $\{x, y, z, u, v, w\} \subset V = \{0, 1, \ldots, 9\}$ and we have a BIB(10, 30, 9, 3, 2) design which contains the 4 blocks of I. We do not lose the BIB property of our design if we replace these 4 blocks with 4 blocks of II (hence the idea of trade off). However, the support and its size may change by trade off. We have used this technique to produce the designs in Table II. Let us give an example. The following design is a BIB(10, 30, 9, 3, 2) design with support size 21.

012	089	259
013	789	259
023	147	267
123	147	267
045	158	349
046	158	349
056	169	357
456	169	357
078	248	368
079	248	368

Consider the following (6, 3) trade of volume 4:

I	II
012	458
045	128
158	024
248	015

Our BIB design contains blocks in set I. If we remove one copy of each of the 4 blocks of I from our design and add one copy of each of the 4 blocks of II we still have a BIB(10, 30, 9, 3, 2) design. However the resulting design will have support size 23. The remaining designs in Table II are also constructed by the trade off technique.

Remark. Our design with support size 29 is nonisomorphic to Parker's design. This can be easily argued since the seven blocks with x, y or z in the Parker design form a BIB(7, 7, 3, 3, 1) design. However, no seven blocks of our design form a sub-BIB design.

We shall close our paper by proving that up to isomorphism there is a unique BIB(10, 30, 9, 3, 2) design based on 21 distinct blocks.

3.4. *The Uniqueness of the* BIB(10, 30, 9, 3, 2) *Design with* 21 *Distinct Blocks*

LEMMA 3.5. *If* $D = E_1 \cup 2E_2$ *is a* $BIB(10, 30, 9, 3, 2)$ *design which has exactly* 21 *distinct blocks, then* E_2 *consists of* 9 *blocks of* (3, 3, 3)*-type.*

Proof. The proof is similar to that of Lemma 3.4. Since $b^* = 21$ then $|E_1| = 12$ and $|E_2| = 9$. Thus we have $9 \times 3 = 27$ positions in 9 blocks of E_2 to be filled by at most 10 varieties. Therefore at least one variety has to appear in 3 blocks of E_2. Now assume the variety 0 appears in 3 blocks of E_2 and the first three blocks of E_2 and E_1 are

$$\bar{B}_1 = (012), \qquad \bar{B}_2 = (034), \qquad \bar{B}_3 = (056)$$

and

$$B_1 = (078), \qquad B_2 = (079), \qquad B_3 = (089),$$

respectively. Again, each of the remaining 6 blocks of E_2 contains at most one of 7, 8 and 9. That is, at least $6 \times 3 - 6 = 12$ positions of the remaining blocks are to be filled by the varieties 1, 2,..., 6. Hence each of these 6 varieties has to appear in 3 blocks of E_2 due to Lemma 3.1. Therefore the three blocks $\bar{B}_1$, $\bar{B}_2$ and $\bar{B}_3$ are of (3, 3, 3)-type. Note that this simply says that if *a variety appears in* 3 *blocks of* E_2 *then all these* 3 *blocks are of* (3, 3, 3)*-type.* As we have shown above, each of the remaining 6 blocks should contain two varieties from 1, 2,..., 6, thus all the remaining blocks are of (3, 3, 3)-type. Hence the result.

THEOREM 3.3. *The* $BIB(10, 30, 9, 3, 2)$ *design with exactly* 21 *distinct blocks is unique up to isomorphism.*

Proof. Assume $D = E_1 \cup 2E_2$ is a BIB(10, 30, 9, 3, 2) design with exactly 21 distinct blocks. From Lemma 3.5, E_2 consists of 9 blocks of (3, 3, 3)-type and E_1 consists of 12 blocks in which one variety appears in 9 blocks and each of the other varieties appears in 3 blocks. Assume 0 appears in 9 blocks of E_1, then the rest of the three blocks of E_1 will be of (3, 3, 3)-type. By Lemma 3.2 and the similar arguments used in the proof of Lemma 3.3, the structure of E_1 is uniquely determined and is isomorphic to

$$
\begin{array}{llll}
E_1: & B_1 = (123) & B_5 = (456) & B_9 = (789) \\
& B_2 = (012) & B_6 = (045) & B_{10} = (078) \\
& B_3 = (013) & B_7 = (046) & B_{11} = (079) \\
& B_4 = (023) & B_8 = (056) & B_{12} = (089)
\end{array}
$$

While the structure of E_2 is also determined and is isomorphic to

$$
\begin{array}{lll}
\bar{B}_1 = (147) & \bar{B}_4 = (248) & \bar{B}_7 = (349) \\
\bar{B}_2 = (158) & \bar{B}_5 = (259) & \bar{B}_8 = (357) \\
\bar{B}_3 = (169) & \bar{B}_6 = (267) & \bar{B}_9 = (368).
\end{array}
$$

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